

Probing hierarchy of temporal correlation requires either generalised measurement or nonunitary evolution

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Temporal steering and violation of the Leggett-Garg inequality are two different ways of probing the violation of macro-realistic assumptions in quantum mechanics. It is shown here that under unitary evolution and projective measurements the two types of temporal correlations lead to similar results. However, their inequivalence may be exhibited if either one of them is relaxed, i.e., by employing either generalized measurements, or noisy evolution, as we show here using relevant examples.

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I. INTRODUCTION

There exist several counter intuitive phenomena in quantum theory without parallel in classical mechanics, going beyond mere theoretical curiosity to the realm of useful practical applications. Different types of entanglement have been used as resources in tasks such as secure key generation [1], random number generation [2], and distinguishing between quantum channels [3]. Bell-nonlocality and Einstein-Podolsky-Rosen (EPR) steering are two types of quantum correlations that may be present in spatially correlated entangled systems. Violation of Bell-CHSH inequalities [4, 5] prohibits the local hidden variable description of correlations in measurement outcomes corresponding to the two entangled parties. Steering, originally introduced by Schrodinger [6] in response to the EPR paradox [7], is the ability of one party, Alice, to affect the state of another remote party, Bob, through her choice of measurements. Such an ability depends on the entanglement of the pair shared between Alice and Bob, as well as the measurement settings chosen for each particles of the pair.

The concepts of Bell-nonlocality and EPR steering have both been reformulated in terms of information theoretic tasks [8] leading to characterization of entanglement between two parties under various levels of trust assigned to the measurement devices possessed by them. Such a characterization has applications in the secret key generation protocols, i.e., security through violation of Bell-inequality in device independent quantum key distribution (QKD) [9], and through steering in one-sided device-independent QKD [10]. The hierarchy of spatial correlations in quantum mechanics is thereby revealed, i.e., the set of all states violating Bell-CHSH inequalities form a strict subset of the set of all states that are steerable, which in turn themselves form a strict subset of the set of all entangled states [8].

The subject of temporal correlations in quantum mechanics has attracted increased attention in recent years. Temporal correlations refer to the correlations in the out-

comes of measurements performed on the same particle acquired through two or more successive measurements, as distinct from spatial correlations which pertain to correlations in outcomes of measurements performed on two or more spatially separated particles. Following the seminal work of Leggett and Garg [11], various studies have been performed leading to the formulation of macrorealistic inequalities for temporally correlated systems. The Leggett-Garg inequalities (LGI) have been generalized for different physical systems [12]. A bound for temporal correlation has been derived [13] by considering a general scenario whose special cases arise in the context of macro-realism and noncontextuality. The issue of how classicality may emerge for temporally correlated systems have also been studied [14–17]. It has been further shown [18] that necessary and sufficient conditions for macro-realism emerge when no signalling in time (NSIT) is satisfied for all combinations of sequential measurements.

Several experimental results on violations of LGI have been reported [19], signifying the untenability of macro-realistic principles at the quantum level, parallel to the refutation of local realism through experimental violations of Bell-CHSH inequalities. On the other hand, the notion of steering has been extended recently in the domain of temporal correlations through the formulation of single system steering where measurements on a single system are considered at different times [20]. The obtained temporal steering inequality is related to the security bound of the Bennett-Brassard 1984 [21] protocol of QKD. It has been shown [20] that the temporal steering inequality is formally mathematically equivalent to an inequality for spatial steering.

Quantum invasiveness of measurement is ingrained in the above two different formulations of temporal correlations probed through the LGI and the temporal steering inequality. There are three alternative conditions for probing the violation of macrorealism for temporal correlations, viz., through the violation of LGI, NSIT, or WLGI (Wigner type LGI). A comparative study of these alternative conditions has been done in Ref. [16]. Tem-

poral steering on the other hand, is probed through the violation of a weaker notion of the non-existence of a hidden state model (HSM) for the post-measurement state. Though a hidden variable model (HVM) trivially follows from any HSM, it is nontrivial to show how these different notions of temporal correlations differ in real testable situations. There exists no study in the literature along these lines.

Therefore, a natural question arises here as to what, if any, is the relation between these two apparently different ways of probing quantum invasiveness. The motivation for the present work is to probe the similarities and differences pertaining to temporal correlations formulated in the above two ways. To this end in the present work we first describe our protocol of temporal steering and derive a temporal analogue of the CHSH type steering inequality [22]. We then show that steerable and LGI violating correlations are equivalent in the context of unitary time evolution and projective measurements.

The degeneracy between the two types of temporal correlations pertaining to the violation of LGI on the one hand, and single system steering on the other, is thus evident at the level of unitary evolution and standard projective measurements. In order to break this degeneracy, we next consider more general measurements. The hierarchy between the two different kinds of temporal correlations may be revealed by employing generalised quantum measurement dubbed as positive operator valued measurements (POVM), as we conjecture through the help of a relevant example. The dissimilarity with the case of spatial correlation is apparent since a only projective measurement is sufficient for differentiating Bell nonlocality and spatial steerability. We finally consider the case of non-unitary evolution with projective measurements, which demonstrates that the LGI violating correlations are stronger than those that allow temporal steering.

II. DIFFERENT KINDS OF TEMPORAL CORRELATIONS

We begin by describing the scenario of temporal correlations of a single system undergoing time evolution. Temporal correlations have been considered in a general form in Ref. [13] irrespective of the type of evolution or compatibility of the observables sequentially measured. Measurement of commutative observables invokes a scenario of quantum contextuality test, whereas noncommutative observables correspond to a test of macrorealism. Here we consider different possible types of temporal correlations analogously to the case of different spatial correlations.

Protocol of single system steering: Sequential measurements on a single system is usually done by a single observer at different times. It can also be thought as if

there are two observers, say, Alice and Bob measuring on a single system sequentially. In our steering protocol in a single run Alice measures on the system first according to Bob's request, and then sends it to him via some quantum channel, announcing her result publicly. Finally, Bob measures on the system in his possession. A system evolves from an initial state $\rho(t=0)$ to some state $\rho(t)$ at time t under some quantum channel. After performing a general measurement (POVM) on the system as asked by Bob, Alice sends it to him via a quantum channel. For example, Alice can create an assemblage like $\{\rho(a_k|k)\}$ according to her choice of measurement $\{M^k(a_k)\}$. Here a_k denotes the outcome a of the k -th POVM and $k \in \{1, 2, \dots, n\}$; $a \in \{0, 1, \dots, d\}$. The set of POVMs satisfies $M^k(a_k) \geq 0$ and $\sum_a M^k(a_k) = \mathbb{I}$. The probability of getting a_k is given by $p(a_k) = \text{tr}[\rho M^k(a_k)]$.

Bob needs to verify whether Alice gave him a post-measured assemblage $\{\rho(a_k|k) = \sqrt{M^k(a_k)}\rho\sqrt{M^k(a_k)}\}$ or not. If Bob finds after state tomography that the assemblage $\{\rho(a_k|k)\}$ has come from a fixed ensemble $\{p(\lambda), \rho_\lambda\}$, called hidden state ensemble (HSE), (where $p(\lambda)$ is the distribution of states ρ_λ), he is not convinced about Alice's steerability. In this case Alice can adopt a classical strategy (cheating strategy) to prepare the state for Bob, which can be written in terms of HSE as

$$\rho(a_k|k) = \sum_\lambda p(\lambda) p(a_k|\lambda, k) \rho_\lambda \quad (1)$$

This means Alice has send Bob states by picking from the HSE with associated weights, where $0 \leq p(a_k|\lambda, k) \leq 1$, $\sum_a p(a_k|\lambda, k) = 1$. Alice can not steer implies that Alice's measurements have no effect on the system at the other side. Mathematically, $\rho^B(M^1) = \rho^B(M^2) = \sum_a \rho(a_k|k)$, where ρ^B is the nonselective postmeasurement state at Bob's hand, and M^k are measurements done by Alice.

Now let us see how joint probabilities for sequential measurements can be constructed according to the classical strategies and indicate some quantum information processing tasks outperforming such classical strategies. Suppose Alice measures A_1 or A_2 at an earlier time t_A on the system, and Bob measures B_1 or B_2 at a later time t_B . Here, three types of possibilities can arise:

(i) In the *non-invasive realist model* (NIRM), a hidden variable model (HVM) pertinent to the LG scenario, the joint probabilities can be written as

$$P(A_i = a_i, B_j = b_j) = \sum_\lambda p(\lambda) p(a_i|A_i, \lambda) p(b_j|B_j, \lambda) \quad (2)$$

This means Alice and Bob obtain their outcome a_i, b_j when measuring A_i, B_j according to some predetermined strategy λ . This NIRM (2) leads to an LGI [23]. Quantum violation of this inequality has been linked with information processing tasks such as saving memory in computing [22], in the context of QKD [24] and random-

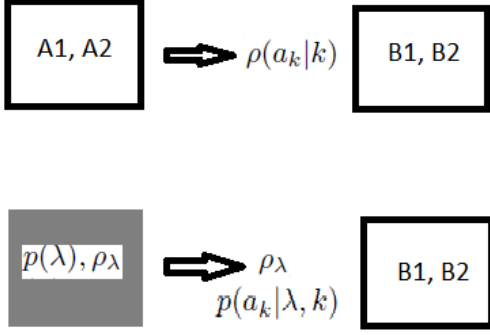


FIG. 1: Schematic diagram for single system steering when each party measures from different sets of two observables. (i) Alice measures A_1 or A_2 according to the request from Bob, sends the post-measurement state to him through some quantum channel and announces her outcome publicly. (ii) Classical mimicry of the above case. Alice picks ρ_λ with probability p_λ , announces outcome a_k with probability $p(a_k|\lambda, k)$ and sends the particle to Bob.

ness generation [25]. As only classical communication between two parties can simulate violation of LGI, it should be emphasised that fully device independent information processing tasks can not be devised with temporal correlations without any other restrictions.

(ii) There exists a hidden state model (HSM) for Bob when Alice is not capable of steering, and joint probabilities (equivalent to eq.(1)) can be written as

$$P(A_i = a_i, B_j = b_j) = \sum_{\lambda} p(\lambda) p(a_i|A_i, \lambda) p^Q(b_j|B_j, \rho_\lambda) (3)$$

Violation of any inequality derived from eq.(3) is a demonstration of temporal steering. This enables secure key generation under coherent attack by cloning [26]. Some semi-device independent tasks can be formulated in this case but we do not focus on this issue here.

(iii) In the case when it is known that Alice sends quantum systems for Bob's measurement, they can always generate a secure key relying on the coherence property of the state and the uncertainty relation according to the original BB84 protocol [21]. It may be mentioned here that entanglement in space is described by the tensor product structure between states belonging to different Hilbert spaces corresponding to spatially separated systems. However, time evolution leads to the transformation of states in the same Hilbert space. Hence, it is not possible to describe states separated in time by a similar tensor product structure, and correspondingly, there exists no analogous separability criterion for temporal correlations.

III. EQUIVALENCE UNDER UNITARY EVOLUTION AND PROJECTIVE MEASUREMENTS

In spite of the obvious differences in the joint probability distributions given by Eqs.(2) and (3) for the two distinct cases (i) and (ii), respectively, we now show that for maximally mixed initial state of any dimension evolving unitarily, they are equivalent under projective measurement. To prove this we introduce a lemma of optimal ensemble for the temporal scenario adapted from an analogous result for the spatial case [8]. No other ensemble can satisfy Eq.(1) iff the optimal ensemble cannot satisfy it.

Lemma 1: Consider a group G of unitary representation $U(g)$ on the Hilbert space of the system. Suppose $\forall A \in \mathcal{M}_A$ (which Alice can measure), $\forall a$ and $\forall g \in G$, if we have $U_1(g)^\dagger A U_1(g) \in \mathcal{M}_A$ and $\tilde{\rho}_a^{U_1(g)^\dagger A U_1(g)} = U_2(g) \tilde{\rho}_a^A U_2(g)^\dagger$, then there exists a G -covariant optimal ensemble: $\{\rho_\lambda^*, p_\lambda^*\} = \{U_2 \rho_\lambda^* U_2^\dagger, p_\lambda^*\}$.

Proof: Suppose a HSM $\{p_\lambda, \rho_\lambda\}$ exists such that $\tilde{\rho}_a^A = \sum p_\lambda \rho_\lambda p(a|A, \lambda)$. Then, from the condition of lemma we have $U_2 \tilde{\rho}_a^A U_2^\dagger = \sum p_\lambda^* U_2 \rho_\lambda U_2^\dagger p(a|A, \lambda)$ and $\tilde{\rho}_a^{U_1^\dagger A U_1} = \sum p_\lambda \rho_\lambda p(a|U_1^\dagger A U_1, \lambda)$. From these we can derive the G -covariant optimal ensemble $F^* = \{p_\lambda^*, U_2 \rho_\lambda U_2^\dagger\}$ with $p_\lambda^* = p_\lambda d\mu_{G(g)}$ with the choice $p(a|A, \lambda) = p(a|U_1^\dagger A U_1, \lambda)$.

Now we are in a position to prove a theorem. In the context of the maximally mixed initial state, unitary evolution and projective measurements, the Lemma 1 holds. As the maximally mixed state is unitarily invariant, we can take the set of all pure states, $\{|\lambda\rangle \in C^d | \langle \lambda | \lambda \rangle = 1\}$, for modelling the HVM, and the probability measure as the Haar measure over the unitary group. (For construction of HVM for $U \otimes U$ invariant bipartite states see [27].)

Theorem 2: For the initial maximally mixed state under unitary evolution and projective measurements the LGI and TSI violating correlations are equivalent.

Proof: If there is a HSM for temporal correlation then it must also have a HVM, i.e. $\text{HSM} \Rightarrow \text{HVM}$. This follows trivially since Eq.(3) resembles Eq.(2) by simply denoting $p(b_j|B_j, \lambda) = p^Q(b_j|B_j, \lambda)$.

Now we show for projective measurements and unitary evolution that the converse of the above implication, i.e., $\text{HVM} \Rightarrow \text{HSM}$ is also true (TSI violation implies violation of LGI). To this end it is sufficient to show that if a HVM exists, the states at Bob's hand can be written through a HSE, or in other words, Alice can simulate $\tilde{\rho}_a^A$ using the HSE $\{p_\lambda, \rho_\lambda\}$ with the same $p_\lambda, p(a|A, \lambda)$ appearing in the HVM. In the steering protocol Bob asks Alice to measure A , and after measuring she announces the outcome a . Then Bob gets

the unnormalised state $\tilde{\rho}_a^A = p(a|A)\rho_a^A$, where ρ_a^A is the eigenstate of A with eigenvalue a . Now, using the Lemma 1, the optimal ensemble exists, and without loss of generality let it consist of pure states with the Haar measure $\{\rho_\lambda^*, p_\lambda^*\}$, s.t. $\tilde{\rho}_a^A = \sum p_\lambda^* \rho_\lambda^* p^*(a|A, \lambda)$. From the existence of the HVM, we have $p(a|A) = \sum p_\lambda p(a|A, \lambda)$, and since the optimal ensemble exists, we must have also $\sum p_\lambda p(a|A, \lambda) = \sum p_\lambda^* p^*(a|A, \lambda)$. As ρ_λ and ρ_λ^* are pure states, they are unitarily related and the invariance of the Haar measure over all spherical rotations implies $p_\lambda = p_\lambda^*$. Consequently, we have $p(a|A, \lambda) = p^*(a|A, \lambda)$. Hence, the theorem. \square

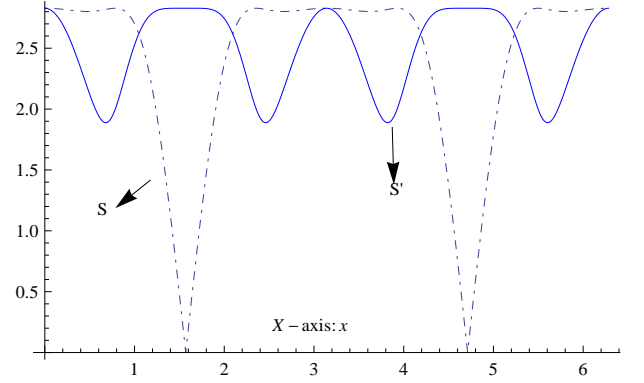
Now we present an example of a single qubit system showing that for the above mentioned conditions LGI and TSI are both violated. Note first, that it is possible to derive a *temporal CHSH inequality* [22] when the two observers Alice (Bob) choose to measure between their corresponding dichotomic observables A_1 and A_2 (B_1 and B_2) at times t_A (t_B) respectively. It has been shown under general conditions that for dichotomic measurements the maximal value for LGI is obtained on qubits [13]. Hence, it suffices to consider qubit systems as far as dichotomic observables and the maximal value of LGI are concerned. With these measurements the temporal CHSH inequality derived in analogy with spatial one is given by [22]

$$|E(A_1, B_1) + E(A_1, B_2) + E(A_2, B_1) - E(A_2, B_2)| \leq 2, \quad (4)$$

where the two time correlation functions are given by $E(A_i, B_j) = p(a_i = b_j) - p(a_i \neq b_j)$. Recently, a steering inequality for spatial correlations has been derived [28] that is analogous to the CHSH inequality in the Bell-nonlocality scenario, in the sense that it forms a necessary and sufficient condition for the case of two parties and dichotomic measurements. The inequality for spatial correlations obtained in Ref. [28] can be derived also in the scenario of temporal correlations involving single system steering using the joint probability distribution (3) (in a similar way as the joint probability distribution (2) leads to a temporal analogue of the CHSH inequality or the LGI). Thus, the following inequality holds when Alice prepares the state for Bob from a HS ensemble:

$$S = \sqrt{\langle (A_1 + A_2)B_1 \rangle^2 + \langle (A_1 + A_2)B_2 \rangle^2} + \sqrt{\langle (A_1 - A_2)B_1 \rangle^2 + \langle (A_1 - A_2)B_2 \rangle^2} \leq 2 \quad (5)$$

The usual LG scenario with unitary evolution of the form $U(t) = e^{-iHt/\hbar}$, and measurements of the same observable (say Q) at different times can be described as two sequential measurements in a row in the following way. For the choices of measurements $A_1 = Q(t_1 = \delta t)$, $A_2 = Q(t_3 = 3\delta t)$, $B_1 = Q(t_2 = 2\delta t)$, $B_2 = Q(t_4 = 4\delta t)$, where $Q(t) = \sigma_z \cos \omega t + \sigma_x \sin \omega t$, the steering function S defined in Eq.(5) is given



by $S = \frac{\sqrt{5 \cos^2 x + \cos^2 3x + 2 \cos 3x \cos x}}{\sqrt{\cos^2 x + \cos^2 3x - 2 \cos 3x \cos x}} + \frac{\sqrt{5 \cos^2 x + \cos^2 3x - 2 \cos 3x \cos x}}{\sqrt{\cos^2 x + \cos^2 3x + 2 \cos 3x \cos x}}$. Here $x = \omega\delta t$, with δt being the time difference between two successive measurements taken to be the same for all pairs of successive measurements. Now, permuting the times t_1 and t_4 in Eq.(5), we get $S' = \frac{\sqrt{\cos^2 x + 2 \cos^2 2x + \cos^2 3x + 2 \cos 2x(\cos x + \cos 3x)}}{\sqrt{\cos^2 x + 2 \cos^2 2x + \cos^2 3x - 2 \cos 2x(\cos x + \cos 3x)}} + \frac{\sqrt{\cos^2 x + 2 \cos^2 2x + \cos^2 3x - 2 \cos 2x(\cos x + \cos 3x)}}{\sqrt{\cos^2 x + 2 \cos^2 2x + \cos^2 3x + 2 \cos 2x(\cos x + \cos 3x)}}$. In Fig.2 we plot the two steering sums S and S' versus the parameter x . It can be seen from Fig.2 that at least one of the steering sums is greater than the no-steering bound, and hence, the steering inequality (5) is violated for any value of x .

Now we show with the above choice of measurements, that an LGI is also violated for any choice of time interval between successive measurements. The inequality (4) is violated maximally for $A_1 = \frac{B_1+B_2}{\sqrt{2}}$ and $A_2 = \frac{B_1-B_2}{\sqrt{2}}$. These measurements can be mapped as, $Q(\omega t = 0) = A_1 = \sigma_z$; $Q(\pi/4) = B_1 = (\sigma_z + \sigma_x)/\sqrt{2}$; $Q(\pi/2) = A_2 = \sigma_x$; and $Q(3\pi/4) = B_2 = (\sigma_x - \sigma_z)/\sqrt{2}$. With permutation of the time indices there exist three different 4-term LGIs and as shown in [29], there is violation of at least any one of these inequalities for an arbitrary choice of the time interval of successive measurements. With sharp measurement under unitary evolution, for arbitrary time intervals of successive measurements ' δt ', whenever there is violation of a TSI, there is violation of a LGI corresponding to the same initial state, and the converse follows trivially.

IV. PROBING HIERARCHY OF TEMPORAL CORRELATIONS

A. Unitary evolution and generalised measurement

We now consider generalised measurements (POVMs) on the system undergoing unitary evolution in order to

show that the two types of temporal correlations characterised by the violation of TSI and LGI respectively, are inequivalent in this scenario. It is possible to consider joint measurement of a set of POVMs even when they do not commute. A set of POVMs $M^k(a_k)$ is said to be compatible if their outcome statistics can be found as marginal of a global POVM $\{G(\lambda); G(\lambda) \geq 0; \sum_{\lambda} G(\lambda) = 1\}$ statistics. Here $\lambda = (a_1, a_2, \dots, a_n)$ and $M^k(a_k) = \sum_{i \neq k} G(\lambda)$. From this global POVM marginal statistics can be obtained through classical post processing of grand statistics [30], $p(a_k) = \sum_{\lambda} g(\lambda) p(a_k|\lambda)$, $g(\lambda) = \text{tr}[\rho G(\lambda)]$. For the case of spatial correlations it has been shown [31] that if Alice's observables are jointly measurable, the joint statistics can be reproduced by a local model for any bipartite state and any measurement of Bob. Moreover, for any set of POVMs at Alice's side that is not jointly measurable, there exists a bipartite state and a set of measurements for Bob such that the resulting joint statistics violates a Bell inequality. For the case of two dichotomic POVMs non-joint measurement is necessary and sufficient for demonstrating steering and Bell nonlocality [32]. However, for three dichotomic POVMs not satisfying full but with pair wise joint measurability, there is no violation of a large class of Bell type inequalities, whereas steering can be shown for this scenario [31].

For temporal correlations with two dichotomic measurements it has been shown [16, 23] that jointly measurable observables can not lead to LGI violation. Necessity and sufficiency of nonjoint measurability to demonstrate temporal steering has been demonstrated [33]. Here we find that there exist nonjoint measurable observables that can demonstrate steering without leading to LGI violation. A quadratic steering inequality for measurements in $N = 2$ or 3 mutually unbiased basis is given by [20]

$$S_N = \sum_{i=1}^N E[\langle B_i \rangle_{A_i}^2] \leq 1. \quad (6)$$

where, $E[\langle B_i \rangle_{A_i}^2] = \sum_{a_i=\pm 1} p(A_i = a_i) \langle B_i \rangle_{A_i=a_i}^2$, with $p(A_i = a_i)$ being the probability of getting a_i at t_A , and $\langle B_i \rangle_{A_i=a_i}^2$ is the expectation value of B_i at t_B on the state measured by Alice at t_A .

Let us consider three dichotomic POVMs acting on the two dimensional Hilbert space as $M^k(a_k) = \frac{1}{2}(\mathbb{I} + \eta a_k \sigma_k)$. This is an example of an unsharp measurement with sharpness parameter η [34], where $k \in \{x, y, z\}$, and σ_k are Pauli matrices. We make the following choice of the observables: $A_1 = \eta \sigma_z$, $A_2 = \eta \sigma_y$, and $A_3 = \eta \sigma_x$. The system evolves under the Hamiltonian $U = e^{-i\sigma_x \omega t/2}$ when A_1, A_2 are measured and $V = e^{-i\sigma_y \omega t/2}$ when A_3 is measured. Going to Heisenberg picture, Bob's observables are given by $B_{1(2)} = U^\dagger \sigma_z (\sigma_y) U$, and $B_3 = V^\dagger \sigma_x V$. With the above choices we get $S_3 = 3\eta^2 \cos^2 \theta$. It is now straightforward to see that Eq.(6) is violated for $\eta > \frac{1}{\sqrt{3}} (= 0.57735)$.

We now consider a class of LGI is given by (see Ref. [29] and references therein)

$$K_n = C_{21} + C_{32} + \dots + C_{n(n-1)} - C_{n1} \quad (7)$$

where, K_n is the n -term LG sum derived from outcome statistics of measurements of an observable, Q at times t_1, t_2, \dots, t_n , and C_{ij} is the correlation between two sequential measurements. Under macrorealist assumptions this quantity is bounded by $-n \leq K_n \leq n-2$; $n \geq 3$, for odd n , and by $-(n-2) \leq K_n \leq n-2$; $n \geq 4$, for even n . It has been shown [16] that for $\eta \leq \sqrt{(n-2)/(n \cos \frac{\pi}{n})}$, no violation can be found.

As we want to compare with the three measurement steering scenario, the relevant LGIs are K_5 and K_6 . Both of them can be mapped to a situation where Alice and Bob measure three different observables on the same system at time t_A and t_B sequentially, with no time evolution of the state between the measurements of Alice and Bob. We consider the LGI

$$K_5 = C_{21} + C_{32} + C_{43} + C_{54} - C_{51}. \quad (8)$$

In fact, without loss of generality any higher order LG test on a single system can thus be mapped to a series of two sequential measurements where Alice chooses to measure first and then Bob from different sets of observables. This mapping scenario is different from the one discussed in [35], as in the latter the mapping from spatial correlation to temporal correlation cannot be extended beyond more than two outcomes and two parties. For sequential measurements one can show that for macrorealist and noncontextual theories the bound is the same, i.e., $K_5 \leq 3$ which can be violated in quantum mechanics. It is shown in [13] that a Tsirelson like bound for macrorealist theory is 4.04, whereas for noncontextual theories it is 3.94.

Here, in order to reproduce two point correlations in eq.(8) yielding maximal violation in the mapped situation, Alice's choice of measurements are the three observables $A_1 = Q(\pi/5)$, $A_2 = Q(3\pi/5)$, $A_3 = Q(\pi)$, and Bob's choices are $B_1 = Q(2\pi/5)$, $B_2 = Q(4\pi/5)$, $B_3 = Q(\pi/5)$, where $Q(\theta) = (\sigma_z \cos \theta + \sigma_x \sin \theta)$. The correlation $C_{21} = \langle A_1 B_1 \rangle$ means Alice first measures A_1 and then Bob measures B_1 sequentially on the same system. Similarly for $C_{32} = \langle A_2 B_1 \rangle$, $C_{43} = \langle A_2 B_2 \rangle$, $C_{54} = \langle A_3 B_2 \rangle$, $C_{51} = \langle A_3 B_3 \rangle$. For this kind of mapping to hold it is crucial that for dichotomic observables, C_{ij} s are independent of the order of the measurements as, $C_{ij} = \frac{1}{2} \text{tr}[\rho \{A_i, B_j\}]$ [35].

It turns out that for $\eta \leq 0.861186$, no violation of the LGI is possible in this case. From the previous discussion it is clear that temporal steering is possible for $\eta > 0.57735$ as $S_3 > 1$. Hence, in the range $0.861186 > \eta > 0.57735$ steering can be shown but no LGI violation can be demonstrated. If we consider $K_6 = C_{21} + C_{32} + C_{43} + C_{54} + C_{56} - C_{61}$, Alice's choices

are $A_1 = Q(\pi/6)$, $A_2 = Q(\pi/2)$, $A_3 = Q(5\pi/6)$ and Bob's are $B_1 = Q(\pi/3)$, $B_2 = Q(2\pi/3)$, $B_3 = Q(\pi/6)$. In this case in the range $0.877383 > \eta > 0.57735$ steering can be demonstrated but not LGI violation. In the context of three measurement settings it is sufficient to consider K_5 and K_6 . Even if we consider all the higher order LGIs, it is straightforward to see from the upper bound of η given above, that the range for which steering can be shown but no LGI violation can be demonstrated increases, thus strengthening the inequivalence between them. It should be mentioned here though that we do not prove rigorously the inequivalence. For non-joint measurement we show that a large class of LGIs are not violated, and based on this we conjecture that no violation of LGI can be found.

The hierarchy between the two types of temporal correlation may be generalized to higher dimensional systems as well. To demonstrate temporal steering non-joint POVMs are required, whatever be the dimension of the system or the cardinality of the outcome set of measurements [33]. Temporal steering witness for higher dimensions with nondegenerate measurements has also been derived recently [26]. It is further possible to show that there exist non-joint measurements for which LGI is not violated under the restriction of dichotomic measurements. For spin j systems the parity operator as the dichotomic observable was considered in Ref. [15], and the maximum value of the four term LGI was obtained asymptotically to be 2.481. In Ref. [23] this bound is improved to $2\sqrt{2}$ which is optimal for dichotomic measurements in any dimension by considering a different measurement scheme. In such a scheme [23] the multi-level spin j system is transformed into $(2j+1)(2)/2$ two level systems for $2j+1$ even (odd). Then each system is evolved separately and measured subsequently by the application of operators acting on two dimensional Hilbert spaces. The relevant observable is given by $Q = \frac{\Gamma_z + \Pi}{\sqrt{2j+1}}$, where Π is a null matrix when $2j+1$ is even, and for odd $2j+1$, the only nonvanishing element of Π is $(\Pi)_{N,N} = \frac{1}{\sqrt{2}}$. Here Γ_z is block diagonal matrix with σ_z . Time evolution of the separated two level systems are affected by $U(t) = \exp^{-i\omega t\sigma_x/2} \oplus \exp^{-i\omega t\sigma_x/2} \oplus \dots \oplus \exp^{-i\omega t\sigma_x/2}$. In such a scenario all the treatment of two level systems described above in the present work follows.

B. Sharp measurement under nonunitary evolution

We now show that with sharp measurement under noisy evolution, temporal steering is possible even when the violation of LGI is washed out by noise. Consider a qubit undergoing Rabi oscillation sent through a amplitude damping channel to Bob. The Markovian decay process in Lindblad form is described by

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \frac{\gamma}{2}(2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho + \rho \sigma_+ \sigma_-) \quad (9)$$

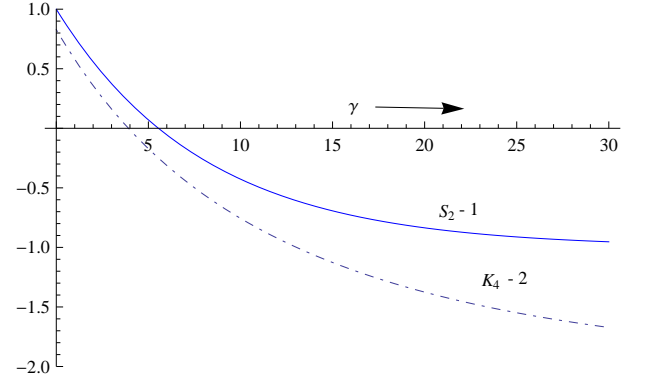


FIG. 3: (Color on-line) The LG sum $K_4 - 2$ (dotted curve) and the temporal steering sum $S_2 - 1$ (solid curve) are plotted respectively versus the damping parameter γ . Positive value of these curves indicates violation of the respective inequalities. Here $\omega\delta t = \pi/4$.

where γ is noise parameter. Taking $H = -\frac{\omega}{2}\sigma_z$ and $\hbar = 1$ with a maximally mixed initial state, the two time correlation for measurement of σ_x is given by $\cos(\omega\delta t) \exp^{-\gamma\delta t}$. The corresponding four term LG sum (K_4 , see eq.(7)) becomes

$$K_4 = 3 \exp^{-\gamma\delta t} \cos(\omega\delta t) - \exp^{-3\gamma\delta t} \cos(3\omega\delta t) \leq 2. \quad (10)$$

For Alice and Bob's choice of measurement given by $A_1 = \eta\sigma_z$, $A_2 = \eta\sigma_y$, $B_1 = U^\dagger\sigma_z U$, $B_2 = U^\dagger\sigma_y U$, the steering parameter S_2 given by eq.(6), becomes

$$S_2 = 2 \exp^{-2\gamma\delta t} \cos^2(\omega\delta t) \leq 1. \quad (11)$$

In Fig. 3 we plot the functions $K_4 - 2$ and $S_2 - 1$ versus the damping parameter γ . It is clear from the figure that after the damping parameter γ exceeds a certain value, the violation of LGI disappears, but temporal steering persists upto a greater value of γ .

V. CONCLUSIONS

In this work we have performed a comparative study of two different kinds of temporal correlations in quantum mechanics: (i) those responsible for single system steering or temporal steering, and (ii) those responsible for the violation of Leggett-Garg inequalities. Any hidden state model gives rise to a hidden variable model, and hence, whenever there is LGI violation there would be TSI violation. The nontrivial result which we obtain here is that when the system evolves under unitary operations and only projective measurements are allowed, the reverse is also true for unitary invariant initial states, *i.e.*, TSI and LGI violating correlations are equivalent.

Next, in order to exhibit the hierarchy between the two types of correlations, we consider again two separate scenarios allowing for either unsharp measurements

(POVMs) or noisy evolution. We find in the former case that there exist non-joint measurements for which steering can be demonstrated whereas violation of a large class of LGIs can not be shown. We conjecture that no violation of LGI can be found in this scenario. This feature may be generalized to arbitrary dimensional systems under the restriction of dichotomic measurements. However, it suffices to consider two level system as far as dichotomic observables are concerned. In the latter case of non-unitary evolution we show that temporal steering is more robust against noise compared to the violation of LGI, again indicating the hierarchy between the two types of correlations.

Before concluding, we note that the hierarchy revealed here for temporal correlations is somewhat analogous to the hierarchy of spatial correlations, but with certain key differences. In the case of spatial correlations the hierarchy between steering and Bell-nonlocality is hidden at the level of pure states, but revealed through the use of mixed states. On the other hand, in the context of temporal correlations, the hierarchy is not at the level of states. Here the hierarchy at the level of correlation between sequential measurements, is hidden if one considers both unitary evolution and sharp measurements, and revealed if either of them is relaxed. It remains for future studies to explore how this hierarchy would fare in the context of multiple outcome measurements. Finally, it may be interesting to formulate protocols for information processing with differential degrees of security or device-independence based on such a hierarchy.

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